MAT 1332, Winter 2007, Assignment #2, Solutions

Total marks=11.

[2] 1. If you simply integrate, the answer is zero. So that obviously isn't right. It's best to sketch the two graphs. See Figure 1.

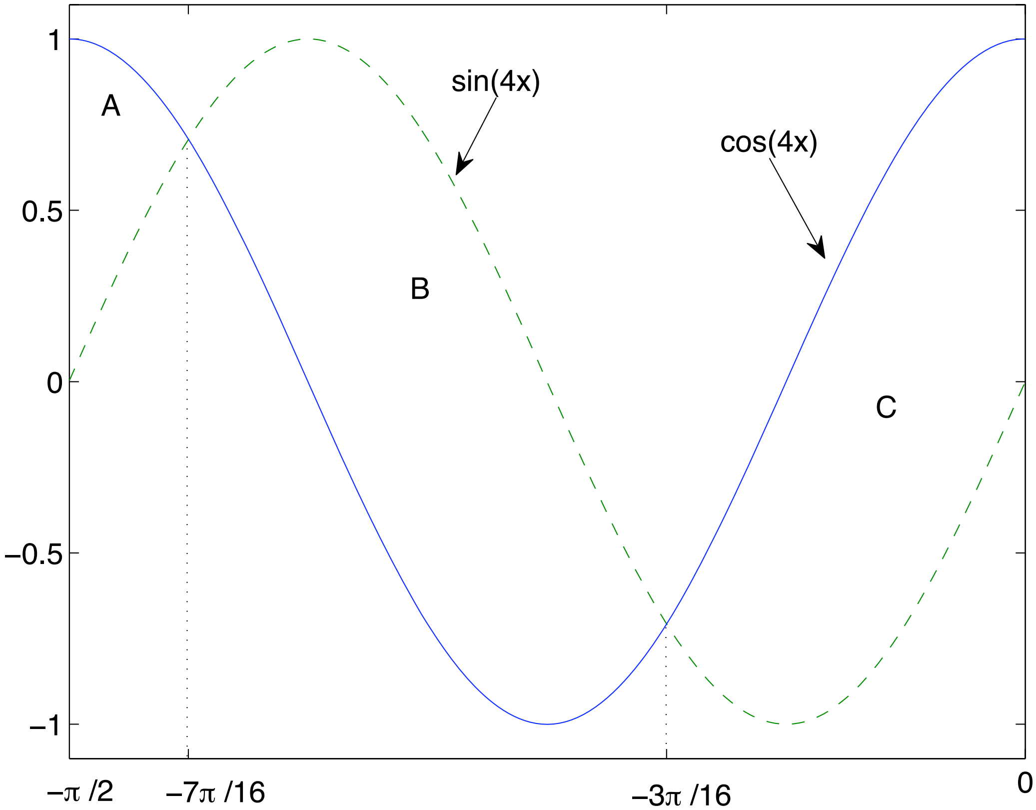

Figure 1: Area between $\sin(4x)$ and $\cos(4x)$.

Clearly the area between them isn't zero. In fact, it's area A, plus area B, plus area C on the graph. And before we can find them, we need to find the points of intersection. That is, the values of x where $\sin(4x) = \cos(4x)$. If you look at Figure 1 again, you may be able to subtly discern these. But let's figure them out:

$$\begin{aligned}
\sin(4x) &= \cos(4x) \\
\tan(4x) &= 1 \qquad\qquad \text{(dividing both sides by } \cos(4x)) \\
4x &= \ldots -\frac{7\pi}{4}, -\frac{3\pi}{4}, \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}\ldots \\
x &= \ldots -\frac{7\pi}{16}, -\frac{3\pi}{16}, \frac{\pi}{16}, \frac{5\pi}{16}, \frac{9\pi}{16}\ldots
\end{aligned}$$

Actually, since $-\frac{\pi}{2} \le x \le 0$, we only need values that fall into this range. So the points of intersection within this range are $x = -\frac{7\pi}{16}$ and $x = -\frac{3\pi}{16}$.

Now we're ready to integrate. The area is the sum of areas A, B and C, so we have

$$\begin{aligned}
\text{Area} &= A + B + C \\
&= \int_{-\pi/2}^{-7\pi/16} (\cos(4x) - \sin(4x))\, dx + \int_{-7\pi/16}^{-3\pi/16} (\sin(4x) - \cos(4x))\, dx + \int_{-3\pi/16}^{0} (\cos(4x) - \sin(4x))\, dx
\end{aligned}$$

$$
\begin{aligned}
&= \left[\frac{\sin(4x)}{4}+\frac{\cos(4x)}{4}\right]_{-\pi/2}^{-7\pi/16}+\left[-\frac{\cos(4x)}{4}-\frac{\sin(4x)}{4}\right]_{-7\pi/16}^{-3\pi/16}+\left[\frac{\sin(4x)}{4}+\frac{\cos(4x)}{4}\right]_{-3\pi/16}^{0}\\
&= \left[\left(\frac{\sin(-7\pi/4)}{4}+\frac{\cos(-7\pi/4)}{4}\right)-\left(\frac{\sin(-2\pi)}{4}+\frac{\cos(-2\pi)}{4}\right)\right]+\left[\left(-\frac{\cos(-3\pi/4)}{4}-\frac{\sin(-3\pi/4)}{4}\right)\right.\\
&\quad \left.-\left(-\frac{\cos(-7\pi/4)}{4}-\frac{\sin(-7\pi/4)}{4}\right)\right]+\left[\left(\frac{\sin(0)}{4}+\frac{\cos(0)}{4}\right)-\left(\frac{\sin(-3\pi/4)}{4}+\frac{\cos(\frac{-3\pi}{4})}{4}\right)\right]\\
&= \left[\left(\frac{1}{4\sqrt{2}}+\frac{1}{4\sqrt{2}}\right)-\left(0+\frac{1}{4}\right)\right]+\left[\left(\frac{1}{4\sqrt{2}}+\frac{1}{4\sqrt{2}}\right)\right.\\
&\quad \left.-\left(-\frac{1}{4\sqrt{2}}-\frac{1}{4\sqrt{2}}\right)\right]+\left[\left(0+\frac{1}{4}\right)-\left(-\frac{1}{4\sqrt{2}}-\frac{1}{4\sqrt{2}}\right)\right]\\
&= \frac{1}{2\sqrt{2}}-\frac{1}{4}+\frac{1}{2\sqrt{2}}+\frac{1}{2\sqrt{2}}+\frac{1}{4}+\frac{1}{2\sqrt{2}}\\
&= \frac{2}{\sqrt{2}}\\
&= \sqrt{2} \;=\; 1.414 \text{ units}^2 .
\end{aligned}
$$

[2] 2. Once again, it's a good idea to draw a sketch. In this case it's a cubic, with positive leading term and the hint has kindly given us the places where it crosses the x-axis, so it's quite easy to draw. For the absolute value part, we turn the negative piece upside down, so it looks like Figure 2.

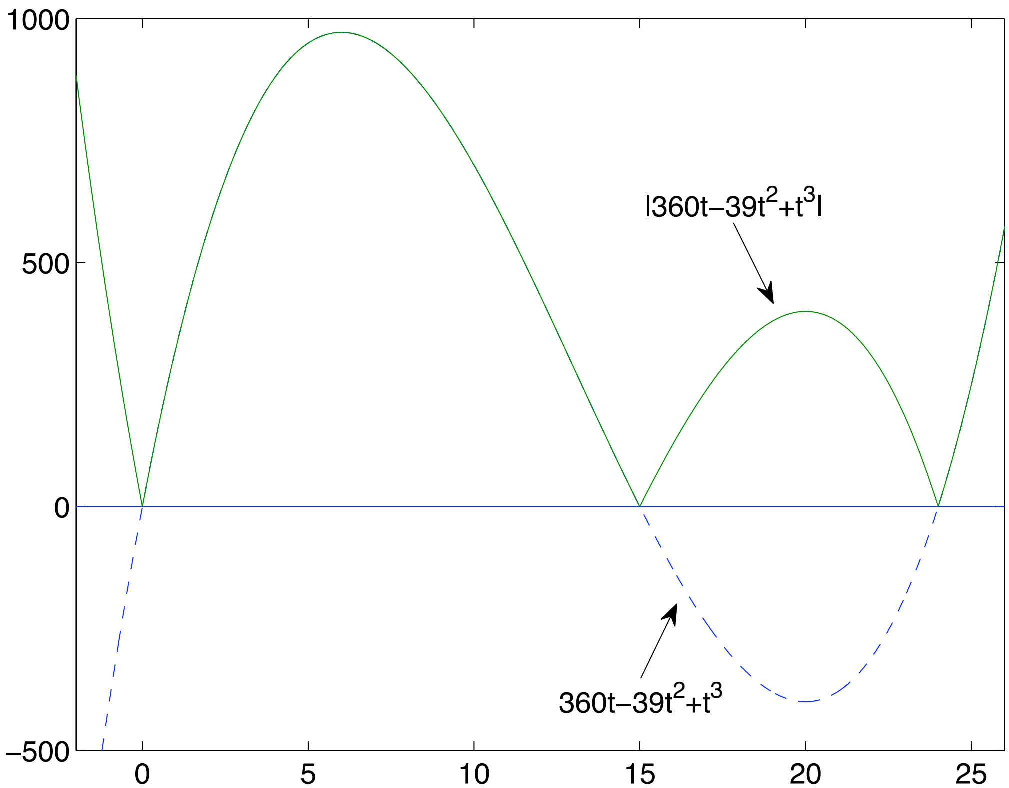

Figure 2: Graph of $|g(x)|$ (solid curve) and $g(x)$ (dotted curve).

Since we have an absolute value, we need to split it into two pieces. The split occurs at $x = 15$, so the first part is the regular (positive) function and the second is the negative of the (negative) function. Thus, the total energy is

$$\text{Total} \;=\; \left[\int_0^{15}\left(360t-39t^2+t^3\right)dt+\int_{15}^{24}\left(39t^2-360t-t^3\right)dt\right]$$

$$
\begin{aligned}
&= \left[180t^2 - 13t^3 + \frac{t^4}{4}\right]_0^{15} + \left[13t^3 - 180t^2 - \frac{t^4}{4}\right]_{15}^{24} \\
&= [9281.25 - 0] + [-6912 - (-9281.25)] \\
&= 11650.5 \text{ joules/hour}\,.
\end{aligned}
$$

3. The derivative is

$$
\begin{aligned}
\rho'(x) &= 4 \times 10^{-8} x(240 - x) - 2 \times 10^{-8} x^2 \qquad \text{(using the chain rule)} \\
&= 9.6 \times 10^{-6} x - 4 \times 10^{-8} x^2 - 2 \times 10^{-8} x^2 \\
&= 9.6 \times 10^{-6} x - 6 \times 10^{-8} x^2\,.
\end{aligned}
$$

The turning points occur when $\rho'(x) = 0$. Thus $x = 0$ and $x = \frac{9.6\times10^{-6}}{6\times10^{-8}} = 160$.

Since $\rho(0) = 1$ and $\rho(160) = 1.04096$, it's clear that the minimum occurs at $x = 0$ and the maximum occurs at $x = 160$. (You could also use the second derivative test here if you like.)

[0.5] (a) Thus, the maximum is 1.04096.

[0.5] (b) The minimum is 1.

[0.5] (c) The maximum occurs at $x = 160$.

[0.5] (d) The total mass is

$$
\begin{aligned}
\text{Total mass} &= \int_0^{200} \rho(x)dx \qquad \text{(remember 2m=200cm)} \\
&= \int_0^{200} \left[1 + 2 \times 10^{-8} x^2(240 - x)\right] dx \\
&= \int_0^{200} \left[1 + 4.8 \times 10^{-6} x^2 - 2 \times 10^{-8} x^3)\right] dx \\
&= \left[x + 1.6 \times 10^{-6} x^3 - 5 \times 10^{-9} x^4)\right]_0^{200} \\
&= 200 + 12.8 - 8 \\
&= 204.8 \text{ g}\,.
\end{aligned}
$$

[0.5] (e) The average mass is just $\frac{204.8}{200} = 1.024$.

[0.5] (f) The graph is shown in Figure 3.

[2] 4. To solve $I = \int \frac{x^3+1}{x^2+3} dx$, first we need to do long division, because the numerator has a higher degree than the denominator. When you divide $x^3 + 1$ by $x^2 + 3$, the result is x, with a remainder of $-3x + 1$. Thus

$$
\begin{aligned}
\frac{x^3 + 1}{x^2 + 3} &= x + \frac{-3x + 1}{x^2 + 3} \\
&= x - \frac{3x}{x^2 + 3} + \frac{1}{x^2 + 3}\,.
\end{aligned}
$$

So we really have three things to integrate. Thus

$$
I = \int x dx - \int \frac{3x}{x^2 + 3} dx + \int \frac{1}{x^2 + 3} dx\,.
$$

The first is quite easy, of course. The second can be done by the substitution $u = x^2 + 3$. The third looks a little bit like the derivative of arctan, only not quite. So we need to move the 3 out. Be careful here though: this will absorb a $\frac{1}{\sqrt{3}}$ inside the square. Thus

$$
I = \int x dx - \int \frac{3x}{x^2 + 3} dx + \frac{1}{3} \int \frac{1}{\left(\frac{x}{\sqrt{3}}\right)^2 + 1} dx\,.
$$

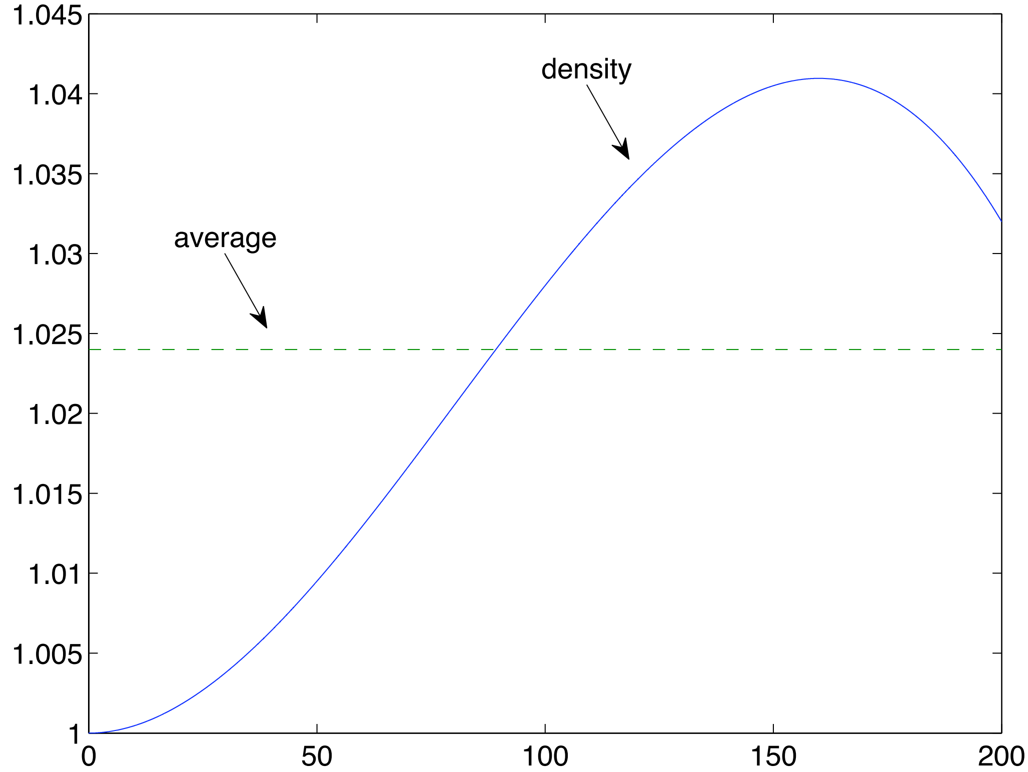

Figure 3: The density (solid curve) and average (dotted curve).

In this last integral, we'll substitute $w = \frac{x}{\sqrt{3}}$. Thus, putting it all together, we have

$$\begin{aligned}
I &= \int x dx - \int \frac{3x}{u}\frac{dx}{2x} + \frac{1}{3}\int \frac{1}{w^2+1}\sqrt{3}dw \\
&= \int x dx - \frac{3}{2}\int \frac{1}{u}dx + \frac{\sqrt{3}}{3}\int \frac{1}{w^2+1}dw \\
&= \frac{x^2}{2} - \frac{3}{2}\ln|u| + \frac{1}{\sqrt{3}}\arctan w + C \qquad\qquad (\text{since } \tfrac{\sqrt{3}}{3} = \tfrac{1}{\sqrt{3}}) \\
&= \frac{x^2}{2} - \frac{3}{2}\ln\left|x^2+3\right| + \frac{1}{\sqrt{3}}\arctan\frac{x}{\sqrt{3}} + C.
\end{aligned}$$

Note: Because of the constant, there are a few other forms of the solution here that are correct, if you manipulate the logarithm. For example,

$$\frac{x^2}{2} - \frac{3}{2}\ln\left|\frac{x^2}{3}+1\right| - \frac{1}{\sqrt{3}}\arctan\frac{x}{\sqrt{3}} + \tilde{C}$$

is an equivalent solution, even though the form of the log term is slightly different.

[2] 5. To solve $J = \int \frac{x^4+3}{x^2-2x-3}dx$, once again the numerator is bigger than the denominator, so we need to do polynomial long division. This leads to

$$\frac{x^4+3}{x^2-2x-3} = x^2+2x+7+\frac{20x+24}{x^2-2x-3}.$$

The first part is easy to integrate, but the second will need to be decomposed via partial fractions. First, we factorise the denominator:

$$\frac{20x+24}{x^2-2x-3} = \frac{20x+24}{(x-3)(x+1)}.$$

Using partial fractions, we write

$$\frac{20x+24}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{x+1}\,.$$

Multiplying both sides by $(x-3)(x-1)$ gives us

$$20x+24 = A(x+1)+B(x-3)\,.$$

Substituting $x=-1$, we have

$$4 = A(0)+B(-4) \quad\Rightarrow\quad B = -1\,.$$

Substituting $x=3$, we have

$$84 = A(4)+B(0) \quad\Rightarrow\quad A = 21\,.$$

Thus

$$\frac{20x+24}{(x-3)(x+1)} = \frac{21}{x-3} - \frac{1}{x+1}\,.$$

Our integral is now

$$\begin{aligned} J &= \int \left(x^2+2x+7+\frac{21}{x-3}-\frac{1}{x+1}\right)dx \\ &= \frac{x^3}{3}+x^2+7x+21\ln|x-3|-\ln|x+1|+C. \end{aligned}$$

6. *Note: This question was not marked.*

The volume obtained by rotating $f(x)=\cos\left(\frac{x}{2}\right)$ around the x-axis, between $-\pi$ and π is given by the formula

$$\begin{aligned} V &= \pi\int_{-\pi}^{\pi}[f(x)]^2\,dx \\ &= \pi\int_{-\pi}^{\pi}\cos^2\left(\frac{x}{2}\right)dx\,. \end{aligned}$$

We can't integrate $\cos^2\theta$ directly, so we have to use our long-forgotten-but-recently-looked-up-in-the-textbook trigonemetric identities. Thus

$$\begin{aligned} \cos 2\theta &= \cos^2\theta-\sin^2\theta \\ &= \cos^2\theta-(1-\cos^2\theta) \qquad\qquad (\text{since } \cos^2\theta+\sin^2\theta=1) \\ &= 2\cos^2\theta-1\,. \end{aligned}$$

Thus

$$\cos^2\theta = \frac{\cos 2\theta+1}{2}\,.$$

Substituting this into the integral, we have

$$\begin{aligned} V &= \pi\int_{-\pi}^{\pi}\frac{\cos 2\left(\frac{x}{2}\right)+1}{2}dx \\ &= \frac{\pi}{2}\int_{-\pi}^{\pi}(\cos x+1)\,dx \\ &= \frac{\pi}{2}\Big[\sin x+x\Big]_{-\pi}^{\pi} \\ &= \frac{\pi}{2}\left[(\sin\pi+\pi)-(\sin(-\pi)-\pi)\right] \\ &= \frac{\pi}{2}(2\pi) \\ &= \pi^2\ \text{units}^3 = 9.8696\ \text{units}^3\,. \end{aligned}$$